Making sense of logarithms in regressions:

The mystery of interpreting logs in regressions can be solved by going back to the lecture notes. There, we were given this crucial relationship:

(FACT 1) *For small changes in x,* $\boldsymbol{\Delta logx} \approx \frac{\boldsymbol{\Delta x}}{\boldsymbol{x}}$ *where ≈ means "approximately equals"*
(Likewise, for small changes in y, $\boldsymbol{\Delta logy} \approx \frac{\boldsymbol{\Delta y}}{\boldsymbol{y}}$ *)*

What does $\frac{\Delta x}{x}$ mean? It is the **proportional** change in x. Example: if $x = 20$ and $\Delta x = 2$, I have $\frac{\Delta x}{x} = \frac{2}{20} = 0.1$, the proportional change in x. We prefer to think of these as **percentage** changes instead. How do I get from 0.1 to the percentage change? Multiply by 100%: $0.1 \times 100\% = 10\%$ change in x. How do I get from a percentage change back to the proportional change? Just divide by 100%: $\frac{10\%}{100\%} = 0.1$. None of this algebra is new to you, but keep it in mind. Formally:

(FACT 2)
$$\frac{\boldsymbol{\Delta x}}{\boldsymbol{x}} = \frac{\textbf{percent change in x}}{\textbf{100}}$$

Now consider the easy-to-interpret "levels" regression:
$$y = \widehat{\beta_0} + \widehat{\beta_1}x$$
We interpret this regression by saying, "ceteris paribus, when x changes by some amount Δx, y changes by $\widehat{\beta_1}(\Delta x)$. For example, if $\widehat{\beta_1} = 2$ and x changes by 3, y changes by $2(3) = 6$. That's the easy part. Formally, this result is:

(FACT 3)
$$\boldsymbol{\Delta y = \widehat{\beta_1}(\Delta x)}$$

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Let's use our FACTs to make sense of the "log" equation: $\boldsymbol{y = \widehat{\beta_0} + \widehat{\beta_1}logx}$

We'll adapt FACT 3 just a little bit to this new equation: $\boldsymbol{\Delta y = \widehat{\beta_1}(\Delta logx)}$

All we did was account for the fact that logx is the independent variable instead of x. Now turning to FACT 1, we switch out $\Delta logx$ for $\frac{\Delta x}{x}$. This gives: $\boldsymbol{\Delta y = \widehat{\beta_1}\left(\frac{\Delta x}{x}\right)}$

Now use FACT 3 to replace $\frac{\Delta x}{x}$ with $\frac{\text{percent change in } x}{100}$. This gives:

$$\boldsymbol{\Delta y = \widehat{\beta_1}\left(\frac{\textbf{percent change in x}}{\textbf{100}}\right)} \textbf{ or } \boldsymbol{\Delta y = \frac{1}{100}\widehat{\beta_1}(\textbf{percent change in x})}$$

This is the interpretation we wanted. Example: If $\widehat{\beta_1}$ is 2, and I change x by 10%, then y changes by 2(10/100) = 2(0.1) = 0.2.

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Now let's apply our FACTs to the log-linear regression: $\boldsymbol{logy = \widehat{\beta_0} + \widehat{\beta_1}x}$

Again, adapt FACT 3 just a bit to get this new equation, just accounting for logy being the dependent variable instead of y: $\boldsymbol{\Delta logy = \widehat{\beta_1}(\Delta x)}$

Use FACT 1 to replace $\Delta logy$ with $\frac{\Delta y}{y}$, to get: $\boldsymbol{\frac{\Delta y}{y} = \widehat{\beta_1}(\Delta x)}$

Use FACT 2 to replace $\frac{\Delta y}{y}$ with $\frac{percent\ change\ in\ y}{100}$ and you get: $\boldsymbol{\frac{percent\ change\ in\ y}{100} = \widehat{\beta_1}(\Delta x)}$

Multiply both sides by 100 to get the interpretation we wanted:

$$\boldsymbol{percent\ change\ in\ y = 100\widehat{\beta_1}(\Delta x)}$$

Example: If $\widehat{\beta_1}$ is 0.3, and I change x by 2, then y changes by 100(0.3)(2) percent = 60 percent.

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Finally, we'll apply our FACTs to the log-log (elasticity) regression: $\boldsymbol{logy = \widehat{\beta_0} + \widehat{\beta_1}logx}$

We follow the same procedure as the last two times. First, apply FACT 3: $\boldsymbol{\Delta logy = \widehat{\beta_1}(\Delta logx)}$

Then, apply FACT 1: $\boldsymbol{\frac{\Delta y}{y} = \widehat{\beta_1}\left(\frac{\Delta x}{x}\right)}$

After that, apply FACT 2: $\boldsymbol{\frac{percent\ change\ in\ y}{100} = \widehat{\beta_1}\left(\frac{percent\ change\ in\ x}{100}\right)}$

Both sides are divided by 100, so let's simplify by cancelling them out:

$$\boldsymbol{percent\ change\ in\ y = \widehat{\beta_1}(percent\ change\ in\ x)}$$

Example: If $\widehat{\beta_1}$ is 0.5, and I change x by 10 percent, then y changes by 0.5(10 percent) = 5 percent.

Recapping what we just did: we took three facts from lecture and combined them to derive, step-by-step, interpretations for all of the regressions involving logarithms. Let's use these interpretations to fill in the worksheet part of **Section Handout 2, part 3**:

We want to see how food consumption (y) measured in \$/year is related to household income (x) measured in \$/year. How would we interpret each of the following models?

Name	Functional Form	Interpretation in Words
linear ("constant returns")	$y = \beta_0 + \beta_1 x + u$	Ceteris paribus, when income increases by _____**z**________, food consumption increases by ______ $\boldsymbol{\beta_1}(\mathbf{z})$ _________.
log ("decreasing returns")	$y = \beta_0 + \beta_1 logx + u$	Ceteris paribus, when income increases by ____ **z percent** ___, food consumption increases by ______ $\frac{1}{100}\boldsymbol{\beta_1}(\mathbf{z})$ _________.
log-linear ("increasing returns")	$logy = \beta_0 + \beta_1 x + u$	Ceteris paribus, when income increases by _____**z**______, food consumption increases by ___ $\mathbf{100}\boldsymbol{\beta_1}(\mathbf{z})$ **percent** _____.
log-log ("constant elasticity")	$logy = \beta_0 + \beta_1 logx + u$	Ceteris paribus, when income increases by ____**z percent**_____, food consumption increases by ______ $\boldsymbol{\beta_1}(\mathbf{z})$ **percent** ______.

Let's do a real example with some numbers. Here I'm going to use different functional forms for regressions relating hourly wage (in \$) with years of education, using Wooldridge's data from example 2.4.

Name	Regression Results	Interpretation in Words
linear ("constant returns")	$\widehat{wage} = -0.90 + 0.54(education)$	When education increases by **1 year**, predicted wage changes by **0.54(1) = \$0.54**.
log ("decreasing returns")	$\widehat{wage} = -7.46 + 5.33log(education)$	When education increases by **10%**, predicted wage increases by $\frac{\mathbf{1}}{\mathbf{100}}\mathbf{5.33(10)}$ = **\$0.533**.
log-linear ("increasing returns")	$\log(\widehat{wage}) = 0.58 + 0.08(education)$	When education increases by **1 year**, predicted wage increases by $\mathbf{100(0.08)(1)\%}$ = **8%**.
log-log ("constant elasticity")	$\log(\widehat{wage}) = -0.44 + 0.83log(education)$	When education increases by **10%**, predicted wage increases by $\mathbf{0.83(10)\%}$ = **8.3%**.